LECTURE 15 RELATED RATES

A review problem on Implicit Differentiation

Before we jump into related rates, we discuss one more problem involving implicit differentiation.

Example 1. Consider $x^2 + xy - y^2 = 1$. Find the equation of the tangent line and normal to the curve at $(2, 3)$.

Solution. To find the slope of the tangent line at $(2, 3)$, we need $\frac{dy}{dx}$ and then evaluate at $(2, 3)$.

$$\begin{aligned} \frac{d}{dx}\left(x^2 + xy - y^2\right) &= \frac{d}{dx}(1) \\ \Longrightarrow 2x + \left(y + x\frac{dy}{dx} - 2y\frac{dy}{dx}\right) &= 0 \\ \Longrightarrow 2x + y + (x - 2y)\frac{dy}{dx} &= 0 \\ \Longrightarrow \frac{dy}{dx} &= -\frac{2x + y}{x - 2y}. \end{aligned}$$

Therefore, the slope at $(2, 3)$ is

$$m = \frac{dy}{dx}\mid_{(2,3)} = -\frac{2 \times 2 + 3}{2 - 2 \times 3} = \frac{7}{4}.$$

The equation for the tangent line then is

$$y - 3 = \frac{7}{4}(x - 2) \Longrightarrow y = \frac{7}{4}x - \frac{1}{2}.$$

The normal is perpendicular to the tangent line at the point of tangency, thus sharing the same point $(2, 3)$. Its slope is the negative reciprocal of the slope of the tangent, which is $-\frac{4}{7}$. Thus, the normal has equation

$$y - 3 = -\frac{4}{7}(x - 2) \Longrightarrow y = -\frac{4}{7}x + \frac{29}{7}.$$

Related Rates

Your textbook has a wonderful introductory example. Suppose you are pumping a spherical balloon. We care about how the volume of the balloon is changing as a function of time.

$$V = \frac{4}{3}\pi r^3$$

is clearly the formula for the volume, but do note that $V = V(t)$ and $r = r(t)$ are both functions of time – as you pump more air, the balloon expands in radius and inevitably volume. Thus, we see that truthfully, the volume satisfies

$$V(t) = \frac{4}{3}\pi r^3(t).$$

Now, by the chain rule, we know

$$V'(t) = \frac{dV}{dr} \cdot \frac{dr}{dt} = 4\pi r^2(t)\, r'(t)$$

which shows that the rate of change of volume is not only dependent on how fast the radius is changing, but also on what the radius is at the moment. In practice, which quantity is easier to measure? $V'(t)$ or $r'(t)$? If you put the hang the expanding balloon on a weight, you can measure its weight right away. As you ensure that the air you pump in is of some constant density (something you can control), you can find the rate of change of volume very easily. Therefore, the above relationship is more helpful to determine $r'(t)$ ($r(t)$ is also easy enough, just measure it).

The foremost point of any related rates problems is to be able to construct a model for the time evolution of some physical quantities. Sometimes, you call for geometry knowledge, such as the volume or surface area formula, while sometimes, you are going to think hard about how to set up the correct relationship from given information. Then, to learn about the relationship about the rate of changes, it is merely an application of the chain rule, which is actually the easier part.

Common relationships that would appear in the current scope of related rates are

(1) Pythagoras – relating lengths to lengths.
(2) Trigonometry – relating angles to lengths.
(3) Volume/Surface Area formula for various shapes – relating volume/surface area to lengths.
(4) Similar triangles – relating lengths to lengths via a common ratio.

So, let's brainstorm for the following modelling problem.

Example 2. Water runs into a conical tank at the rate of 9 ft^3/min. The tank stands point down and has a height of $10ft$ and a base radius of $5ft$. How fast is the water level rising when the water is 6 ft deep?

Solution. Let's look at each sentence.

"Water runs into a conical tank at the rate of 9 ft^3/min."

Looking at the unit, volume per minute, it is definitely related to $\frac{dV}{dt}$.

"The tank stands point down and has a height of $10ft$ and a base radius of $5ft$."

Okay, this tells me the dimension of the cone. This not only helps me calculate the total volume, but also informs me how the water grows via its volume formula.

$$V(t) = \frac{1}{3}\pi r^2(t)\, h(t).$$

where

$$\begin{aligned} V(t) &\quad \text{volume of all water in the tank at time } t, \\ r(t) &\quad \text{radius of the base of water at time } t, \\ h(t) &\quad \text{water level at time } t. \end{aligned}$$

However, we can relate $r(t)$ and $h(t)$ by similar triangles. Note the radius is 5 when the height is 10, then at height h, we should have radius $\frac{h}{2}$. Therefore, our volume equation becomes

$$V(t) = \frac{1}{3}\pi\left(\frac{h(t)}{2}\right)^2 h(t) = \frac{1}{12}\pi h^3(t)$$

"How fast is the water level rising when the water is 6 ft deep?"

Water level, hmmm, must be related to $h(t)$. And the rate at which it is rising? $\frac{dh}{dt}$ – we need to find out what it is. So it seems that we need to have $\frac{dV}{dt} = V'(t)$ and $\frac{dh}{dt} = h'(t)$ somewhere in our process.

Take $\frac{d}{dt}$ of both sides of the volume equation, we have

$$V'(t) = \frac{1}{12}\pi\frac{d}{dt}\left(h^3(t)\right) = \frac{1}{12}\pi 3h^2(t)\, h'(t) = \frac{\pi}{4}h^2(t)\, h'(t)$$

We know, $V'(t) = 9$, a constant rate (at any time), as given. $h(t) = 6$ also as given. Therefore,

$$9 = \frac{\pi}{4}(6)^2 h'(t) \implies h'(t) = \frac{1}{\pi}\, ft/min \approx 0.32\, ft/min.$$

Remark. Moral of the story:

(1) DRAW, DRAW, DRAW A PICTURE. Name the variables.
(2) Write down the numerical information (such as shape dimensions, like radius, height).
(3) Write down what you are asked to find (usually, a rate, so a derivative of some quantity with respect to time).
(4) Write down an equation that relates the variables in step 1. Sometimes, some common sense is needed.
(5) When in doubt, take a derivative.
(6) Evaluate.

More examples involving other equations.

Example 3. A 15-foot ladder is resting against the wall. The bottom is initially10 feet away from the wall and is being pushed towards the wall at a rate of $\frac{1}{4}$ ft/sec. How fast is the top of the ladder moving up the wall 12 seconds after we start pushing?

Solution. The ladder, vertical wall and the floor form a right triangle. The length relationship here is via Pythagoras. Denote floor length and wall height $x(t)$ and $y(t)$ respectively. Then, we see that

$$x^2 + y^2 = 15^2$$

since the hypotenuse is always the same ladder, and thus of length 15. Note that both x and y are actually $x(t)$ and $y(t)$, functions of time.

Now, we want to know how fast the ladder is moving up the wall 12 seconds after we start pushing, so we want the rate of change of "wall height" $y'(12)$. Taking a derivative, we have

$$2xx' + 2yy' = 0 \implies x(t)\,x'(t) + y(t)\,y'(t) = 0.$$

We identify what we know here, as we want to know $y'(12)$.

$$\begin{aligned} x(12) &= 10 - \frac{1}{4} \times 12 = 7. \\ x'(12) &= -\frac{1}{4} \\ y(12) &= \sqrt{15^2 - 7^2} = \sqrt{176} = 4\sqrt{11}. \end{aligned}$$

Plugging back, we find

$$7 \times \left(-\frac{1}{4}\right) + 4\sqrt{11}y'(12) = 0 \implies y'(12) = \frac{7}{4}\frac{1}{4\sqrt{11}} = \frac{7}{16\sqrt{11}} \approx 0.1319\, ft/sec$$